The Stability of Hedonic Coalition Structures

Anna Bogomolnaia and Matthew O. Jackson*

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Abstract

We consider the partitioning of a society into coalitions in purely hedonic settings; i.e., where each player's payoff is completely determined by the identity of other members of her coalition. We first discuss how hedonic and non-hedonic settings differ and some sufficient conditions for the existence of core stable coalition partitions in hedonic settings. We then focus on a weaker stability condition: individual stability, where no player can benefit from moving to another coalition while not hurting the members of that new coalition. We show that if coalitions can be ordered according to some characteristic over which players have single-peaked preferences, or where players have symmetric and additively separable preferences, then there exists an individually stable coalition partition. Examples show that without these conditions, individually stable coalition partitions may not exist. We also discuss some other stability concepts, and the incompatibility of stability with other normative properties.

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*Bogomolnaia is at the Department of Economics, P.O.Box 0496, Southern Methodist University, Dallas, Texas 75275, USA (annab@mail.smu.edu). Jackson is at the division of Humanities and Social Sciences, 228-77, California Institute of Technology, Pasadena, California 91125, USA, (jacksonm@hss.caltech.edu). Financial support under NSF grant SBR 9507912 is gratefully acknowledged. We thank Salvador Barbera, Gabrielle Demange, Joseph Greenberg, Michel Le Breton, and Shlomo Weber for very helpful conversations and suggestions. We are also grateful for the comments of two anonymous referees that have helped in the rewriting of the paper.

1 Introduction

Coalition formation is of fundamental importance in a wide variety of social, economic, and political problems, ranging from communication and trade to legislative voting. As such, there is much about the formation of coalitions that deserves study. In this paper, we examine the purely hedonic aspect of coalition formation. This terminology follows Drèze and Greenberg (1980), who call the dependence of a player's utility on the identity of the members of her coalition the "hedonic aspect". Essentially, the purely hedonic problem that we examine here boils coalition formation down to its purest social form: the payoff to a player depends *only* on the composition of members of the coalition to which she belongs.

Examples of situations where players' preferences in coalition formation are hedonic include the formation of social clubs, groups, and organizations, as well as faculties, teams, and societies. Situations where preferences are derived from activities that a group will undertake, such as the provision of a public good, are also hedonic provided that a player's preferences depend only on the members of her own coalition and not on the composition of other coalitions. For instance, if a coalition median votes on a level of public good to provide, then a player can predict the public good level that will be provided by different coalitions and evaluate coalitions by knowing their membership. More generally, many situations where players form groups and then each group chooses from a set of available alternatives can be reduced to hedonic settings, if reliable predictions can be formed from the beginning about each group's subsequent choice.

While the stability of coalition partitions where players have preferences over members of their coalition has been examined in a number of models (especially those where there are local public goods or some sort of political interaction as in Guesnerie and Oddou (1981), Greenberg and Weber (1986, 1993), Demange (1994), among others),[1] the purely hedonic model covers interesting settings and issues that have not been previously studied.[2] We discuss the differences in more detail in Section 2.

[1] There is also the large literature on the marriage problem of Gale and Shapley (1962), assignment games of Shapley and Shubik (1972), as well as various off-shoots (see Roth and Sotomayor (1990)).

[2] A notable exception to this is Jehiel and Scotchmer (1997). They examine the formation of jurisdictions where voters anticipate a median vote over a level of public good supplied. Their model has a continuum of players and so does not fit into the definition of hedonic settings explored here, but it is still an example of a model where preferences are hedonic. We refer the reader to Jehiel and Scotchmer for some interesting comparisons of various rules for admission into a jurisdiction.

The focus of our paper is on the existence of stable coalition partitions in the hedonic model. While there are some hedonic settings where there exist core stable coalition partitions, as discussed in Section 4,[3] there are many where there do not, and yet there still exist partitions, stable in a non-cooperative sense. The main results of this paper relate to the existence of stable coalition partitions when only individual movements are allowed, so that only one individual considers changing her coalition at a time.

Non-cooperative stability tests makes sense if players are small relative to the size of coalitions or if the cost of coordinating movements to form a new coalition is high. Examples one might have in mind include professors considering changing universities, soccer players considering changing teams, individuals changing communities where their public goods and taxes are decided, and individuals considering changing clubs.[4] One notion that we examine is that of *individual stability*. This concept is based on the concept of "individually stable equilibrium" from a non-hedonic model by Greenberg (1978) and Drèze and Greenberg (1980); but is modified to apply to the purely hedonic setting where no allocations of goods need to be kept track of. A coalition partition is individually stable if it is immune to individual movements which benefit the moving player and do not hurt any member of the coalition she joins.

We begin by showing that if preferences are additively separable and symmetric (i.e., players have the same reciprocal values for each other), then the set of individually stable coalition partitions is nonempty. We also show that with such preferences, the set of Nash stable coalition partitions is nonempty, where Nash stability is a non-cooperative notion of stability that is stronger than individual stability in the sense that players do not need permission to join a new coalition. However, we show that if these conditions on preferences are weakened slightly (for instance, symmetry is weakened to mutuality where any two players have values of the same sign for each other, but not necessary of the same magnitude), then the set of individually stable coalition structures can be empty.

Next, we consider preferences that depend on some underlying summary characteristic of a coalition, and where players have single peaked preferences over these sum-

[3]This same hedonic model is independently examined by Banerjee, Konishi and Sönmez (1998). Through a series of very interesting results and examples, Banerjee, Konishi and Sönmez (1998) show that there are non-trivial settings where core stable coalition partitions exist, but also many natural settings where they do not. We discuss their conditions and some others in Section 4.

[4]These and other examples are discussed by Drèze and Greenberg (1980), to which we direct the reader for additional motivation.

mary characteristics. Examples that fit into the setting we examine include situations where players care only about the size of their coalitions, but not about the identities of the members of the coalition, or where a coalition takes a median vote over a level of a public good to produce and players care only about that choice of the coalition. Somewhat surprisingly, even when preferences are anonymous and single-peaked, the set of Nash stable outcomes can be empty. Nevertheless, these requirements guarantee the existence of an individually stable coalition partition and we provide an algorithm for identifying such a partition. Moreover, we show that the partitions that the algorithm identifies are weakly Pareto efficient as well as individually stable. We go on to show that single-peakedness of preferences is important to the existence of an individually stable coalition partition.

Towards the end of the paper, we also discuss an even weaker notion of stability called *contractual individual stability*, again adapted to the purely hedonic model from a notion of Drèze and Greenberg (1980) that applied to non-hedonic models. We conclude with some examples and remarks regarding other axiomatic properties in hedonic coalition formation, such as strategy-proofness, envy-freeness, and population monotonicity.

2 A Comparison of the Hedonic and Non-Hedonic Settings

It is useful to begin our discussion with a look at an example that highlights the differences between the hedonic and non-hedonic settings, and offers motivation for some of our analysis. It is a standard Tiebout-style local public good model.

A set $N = \{1, \ldots, n\}$ of players is divided into coalitions. Each coalition selects a level of public good to consume. Public good consumption is local and so a player consumers only the public good produced by her coalition. What is feasible for a coalition depends on its size. A coalition $S \subset N$ can produce any amount of public good in $[0, \#S]$. So, each member of the coalition brings a unit of the public good to the coalition and there is free disposal. Let individuals have single-peaked preferences, denoted $\succeq_i$, over the amount of public good that they consume, with each individual's peak lying in $[0, n]$.[5]

[5]For a formal definition of any concept, such as single-peakedness, that is not familiar, see the definition section below.

First, let us consider a standard non-hedonic version of this setting. In the spirit of Greenberg and Weber (1993) and Demange (1994), let an "outcome" be a partition Π of N and a specification of a public good choice $c_S \in [0, \#S]$ for each $S \in \Pi$. An outcome Π, c is core* stable if it cannot be improved upon by any coalition.[6] That is Π, c is core* stable if for any $S' \subset N$ and $c_{S'} \in [0, \#S']$ there exists $i \in S'$ with $c_{S_i} \succeq_i c_{S'}$, where S_i is the element of Π containing i.

If preferences are continuous, then it follows from the results of Greenberg and Weber (1993) (see also Demange (1994)) that there exists an outcome that is core* stable.

Next, let us turn to a hedonic version of the model. Suppose that the manner in which a coalition S chooses c_S is by a median vote of its members over $[0, \#S]$. For simplicity, if there is an even number of voters then choose the lower of the two medians. The model is now purely hedonic: once a coalition is specified then its members can predict the choice of public good that will be selected by the median vote, assuming that players follow their dominant strategies of voting for their most preferred public good level in $[0, \#S]$. So, each player's induced preferences over coalitions depend only on the membership of the coalitions and are thus purely hedonic.

We can now define core stability notion for the hedonic setting. Let $m(S)$ denote the median voting outcome for coalition S over $[0, \#S]$ as described above. We can define preferences over coalitions by saying that $S \succeq_i S'$ if and only if $m(S) \succeq_i m(S')$. A coalition partition Π is core stable in the hedonic model if for every $S' \subset N$ there exists $i \in S'$ such that $S_i \succeq_i S'$, where S_i is the element of Π containing i.

In contrast with the non-hedonic version of the model, there does not always exist a core stable partition, as we show in the following example.[7]

Example 1

Let $n = 7$ and have players' preference peaks over levels of the public good be $p_1 = p_2 = 4$, $p_3 = 5$, and $p_4 = p_5 = p_6 = p_7 = 7$. Also, players 1 and 2 prefer 3 units to 5 units of the public good, and 6 units to 2 units. Player 3 prefers 3 units to 6 units and 6 units to 2 units.

[6] The * on core stability distinguishes this notion from the hedonic one that we will use in this paper.

[7] See Haeringer (2000) for an examination of a related model where players care about both the size and choice of a coalition. He provides conditions on preferences sufficient for the existence of core-like and Tiebout (Nash) stable coalition structures.

In the non-hedonic version of the model, $(\Pi, c) = (\{N\}, 5)$ - all players grouped together and consuming 5 units of the public good, is a core* stable outcome.

In the hedonic version of the model, however, there does not exist a core stable partition. To see this, note that a partition must have a group with least 5 members in order to be core stable or else the partition is blocked by $\{3,4,5,6,7\}$. Any partition that contains a group of 6 or more members and contains all of $\{4,5,6,7\}$ will have the large group produce $m(S) \geq 6$ and will be blocked by $\{1,2,3\}$ which will have $m(S) = 3$. A partition that contains a group of 6 players and leaves one of $\{4,5,6,7\}$ single, will be blocked by the single player together with players 1 and 2. So a core stable partition have to consist of a group of 5 and a group of 2. The partition $\{\{1,2\},\{3,4,5,6,7\}\}$ is blocked by $\{1,2,4,5,6,7\}$. Thus, a core stable partition would have to be of the form $\{S_1, S_2\}$ with $\#S_2 = 5$ and $\{1,2\} \cap S_2 \neq \emptyset$. Without loss of generality, consider a partition where $1 \in S_2$. This is blocked by $S_2 \cup \{1\}$, and so no partition is core stable.

The above example shows that the hedonic model where a coalition's actions are predicted, has different properties than the non-hedonic model.[8] For the Greenberg and Weber allocation to be stable, the coalition must be able to commit not to choose according to median voting. In this example, which of the analyses would be appropriate would depend on the method by which coalitions choose the level of public good.

While there does not always exist a core stable partition in the hedonic version of the model above, there always exists an individually stable partition, as we shall prove below. Individual stability only considers blocking by coalitions that are formed by having one player leave her current coalition and join another coalition in the partition (or move to be single). In the example above, an individually stable partition is $\{\{1,2\},\{3,4,5,6,7\}\}$. Player 3 is worse off if 1 or 2 join the larger coalition, and so closes the larger coalition to their entrance, and no player in the larger coalition wishes to join 1 and 2. Moreover, this partition is Pareto optimal.

[8]To note a further difference between the models, there always exists a Nash stable outcome in the non-hedonic version (which again can be derived from the results of Greenberg and Weber (1993) or Demange (1994)), but one can find hedonic examples where there do not exist Nash stable partitions, as we shall see below.

3 Definitions and notation

The core and Nash stability definitions provided below are based on standard definitions in the literature, while individual stability definitions below are adapted from Greenberg (1978) and Drèze and Greenberg (1980).[9]

Consider a finite set of players $N = \{1, \ldots, n\}$.

A *coalition partition* is a set $\Pi = \{S_k\}_{k=1}^{K}$ which partitions N. Thus, $S_k \subset N$ are disjoint and $\cup_{k=1}^{K} S_k = N$. The subsets S_k are called *coalitions*.

Each individual has preferences over possible coalition partitions which are entirely determined by the coalition that she belongs to. Thus, a player i's *preferences* can be represented by an order $\succeq_i$ (a complete, reflexive, and transitive binary relation) over the set $\{S_k \subset N : i \in S_k\}$. We let $\succ_i$ denote the associated asymmetric binary relation.

Given Π and i, let $S_\Pi(i)$ denote the set $S_k \in \Pi$ such that $i \in S_k$.

A *game* $(N, \succ)$ is set of players and a profile of preferences.

Properties of Preferences:

A player i's preferences are *additively separable* if there exists a function $v_i : N \to I\!R$ such that $\forall S_1, S_2 \ni i$

$$S_1 \succeq_i S_2 \Leftrightarrow \sum_{j \in S_1} v_i(j) \geq \sum_{j \in S_2} v_i(j),$$

where, without loss of generality, we normalize by setting $v_i(i) = 0$.

A profile of additively separable preferences, represented by $(v_1, \ldots, v_n)$, satisfies *symmetry* if $v_i(j) = v_j(i)$, $\forall i, j$.

A profile of additively separable preferences, represented by $(v_1, \ldots, v_n)$, satisfies *mutuality* if $v_i(j) \geq 0 \iff v_j(i) \geq 0$. Thus, symmetry implies mutuality.

A player i's preferences satisfy *anonymity* if $\forall S_1, S_2 \ni i$

$$\#S_1 = \#S_2 \Rightarrow S_1 \sim_i S_2,$$

where $\#S$ denotes the size of a coalition S.

[9] The hedonic model we define here is independently studied by Banerjee, Konishi and Sönmez (1998) who focus on core stability, as discussed in the next section.

A player i's preferences on some set $\{0, 1, \ldots, K\}$[10] are *single-peaked* if there exists a number p_i, called i's *peak*, such that $\forall s_1, s_2 \in \{0, 1, \ldots, K\}$

$$[s_1 < s_2 \leq p_i \text{ or } s_1 > s_2 \geq p_i] \Rightarrow s_2 \succ_i s_1.$$

As an example, i's preferences might depend only on the size of the coalition that i is a member of and might be single-peaked on size.

Efficiency

A coalition partition Π is *weakly Pareto efficient* for any partition $\Pi' \neq \Pi$ there exists a player i such that $S_{\Pi'}(i) \neq S_{\Pi}(i)$ and $S_{\Pi}(i) \succeq_i S_{\Pi'}(i)$.

In this version of weak Pareto efficiency, a coalition partition offers improvement over another if all players whose coalitions change are made strictly better off. The restriction of attention to players whose coalitions have changed makes this definition stronger than the usual definition of weak Pareto efficiency. If players' preferences are strict, then weak Pareto efficiency coincides with Pareto efficiency.

Stability concepts:

A coalition partition Π is *core stable* (or in the *core*) if $\nexists T \subset N$ such that $T \succ_i S_{\Pi}(i)$ for all $i \in T$.

When a coalition partition Π is not core stable, so that $\exists T \subset N$ such that $T \succ_i S_{\Pi}(i)$ for all $i \in T$, we say that T *blocks* Π.

A coalition partition Π is *Nash stable* if $\forall i\ S_{\Pi}(i) \succeq_i S_k \cup \{i\}$ for all $S_k \in \Pi \cup \{\emptyset\}$.

A coalition structure Π is *individually stable* if there do not exist $i \in N$ and a coalition $S_k \in \Pi \cup \{\emptyset\}$ such that $S_k \cup \{i\} \succ_i S_{\Pi}(i)$, and $S_k \cup \{i\} \succeq_j S_k$ for all $j \in S_k$.

In line with individual stability, a coalition is said to be *open* if there is some player that could be added to the coalition without making any of the current members worse off, and a coalition is said to be *closed*, otherwise.

A coalition structure Π is *contractually individually stable* if there do not exist $i \in N$ and a coalition $S_k \in \Pi \cup \{\emptyset\}$, such that $S_k \cup \{i\} \succ_i S_{\Pi}(i)$, $S_k \cup \{i\} \succeq_j S_k\ \forall j \in S_k$; and $S_{\Pi}(i)/\{i\} \succeq_j S_{\Pi}(i)\ \forall j \in S_{\Pi}(i)/\{i\}$.

[10]We are vague about the set underlying preferences in this definition, as we will apply the notion of single-peakedness to several different underlying domains.

Note that individual stability implies individual rationality, since nobody wants to leave her current coalition and stay alone.

The relation between the stability concepts is indicated below, where $\Rightarrow$ indicates that if a partition satisfies the first notion, then it also satisfies the second.

Individual stability $\Rightarrow$ contractual individual stability.

Nash stability $\Rightarrow$ individual stability $\Rightarrow$ contractual individual stability.

However, core stability $\not\Rightarrow$ Nash stability $\not\Rightarrow$ core stability.

Also, core stability $\not\Rightarrow$ individual stability, because our core stability notion is the one of weak core.

The following examples illustrate these relationships.

Example 2 *An undesired guest.*[11]

Let $N = \{1, 2, 3\}$ and

$$\begin{aligned}
&\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\},\\
&\{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \succ_2 \{2,3\},\\
&\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}.
\end{aligned}$$

These orderings can be represented by additively separable utilities. Here, $\{\{1,2\},\{3\}\}$ is in the core and is individually stable, while the set of Nash stable partitions is empty since 3 would like to join with 1 and 2, who would then prefer to be alone.

Example 3 *Two is company, three is a crowd.*

Let $N = \{1, 2, 3\}$ and

$$\begin{aligned}
&\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\},\\
&\{2,3\} \succ_2 \{2,1\} \succ_2 \{1,2,3\} \succ_2 \{2\},\\
&\{3,1\} \succ_3 \{3,2\} \succ_3 \{1,2,3\} \succ_3 \{3\}.
\end{aligned}$$

These preferences have a cycle: the first player prefers the second player to the third, the second player prefers the third player to the first, and the third player prefers the

[11]Drèze and Greenberg (1980) consider a different example by the same name.

first one to the second. All players prefer to be in some couple over being all together, and being alone is the worst outcome. Here, the core is empty, while $\{\{1,2,3\}\}$ is the unique Nash stable partition, as well as being the unique individually stable partition.

Example 4

Let $N = \{1,2,3\}$ and

$$\begin{aligned}
\{1,2\} &\succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3\}, \\
\{2,3\} &\succ_2 \{2,1\} \succ_2 \{2\} \succ_2 \{1,2,3\}, \\
\{3,1\} &\succ_3 \{3,2\} \succ_3 \{3\} \succ_3 \{1,2,3\}.
\end{aligned}$$

This is similar to the previous example except that staying alone is better than being in the grand coalition. Here, there does not exist a core stable, Nash stable, or individually stable coalition partition. Nevertheless, there are three contractually individually stable coalition structures: — $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$ and $\{\{2,3\},\{1\}\}$.

4 Core Stability

Before moving to the main focus of our analysis on individual stability, we provide some idea of what one can say about the existence of core stable partitions.

As mentioned in the introduction, Banerjee, Konishi, and Sönmez (1998) identify two conditions, the top coalition and the weak top coalition property, that are sufficient for the existence of a core stable partition in the hedonic model. In addition to those conditions, one can easily adapt conditions from the NTU cooperative game literature that are distinct from their weak top coalition property. These conditions capture applications, such as multi-sided matching problems, that are not captured by the weak top coalition property. We state each of these conditions and then show that they are all distinct.

In order to state the first two conditions, note that the hedonic setting can be thought of as an NTU game where the allocations of a coalition are unique. This is more formally stated as follows:

Consider a hedonic game $(N, \succ)$. Choose a profile of utility functions $\{u_i\}_{i \in N}$ on the set of coalitions such that $u_i(S) > u_i(T)$ if and only if $S \succ_i T$. Define the following NTU game. For each $S \neq N$ let

$$V(S) = \{x \in I\!R^n : x_i \leq u_i(S) \ \forall i \in S\},$$

and let

$$V(N) = \{x \in I\!R^n : \exists \Pi \text{ s.t. } x_i \leq u_i(S_\Pi(i)) \ \forall i\}.$$

It is clear that the core of this NTU game[12] is nonempty if and only if there exists a core stable coalition partition for $(N, \succ)$.

With this observation, one can adapt conditions from the NTU setting to the hedonic setting.

Ordinal Balance

The first condition builds directly on a theorem of Scarf (1967).

A collection of coalitions $\mathcal{B}$ is *balanced* if there exists a vector of positive weights d_S, such that for each player $i \in N$ $\sum_{S \in \mathcal{B}: i \in S} d_S = 1$.

A game $(N, \succ)$ is *ordinally balanced* if for each balanced collection of coalitions $\mathcal{B}$ there exists a coalition partition Π such that for each i there exists $S \in \mathcal{B}$ with $i \in S$ such that $S_\Pi(i) \succeq_i S$.

Thus, a game is ordinally balanced if for each balanced family of coalitions there exists some coalition partition such that each player prefers her coalition in the partition to her worst coalition in the balanced family.

Consecutiveness

The following definitions are adaptations of corresponding definitions for NTU games by Greenberg and Weber (1986) and Greenberg (1994).

An *ordering of players* is a bijection $f : N \to N$.

A coalition $S \subset N$ is *consecutive with respect to an ordering* f, if $f(i) < f(j) < f(k)$, $i \in S$, and $k \in S$ imply $j \in S$.

A game $(N, \{\succeq_i\}_{i \in N})$ is *weakly consecutive* if there exists an ordering of players, f, such that whenever Π is defeated by some T, there exists T' that is consecutive with respect to f that defeats Π.

A game $(N, \{\succeq_i\}_{i \in N})$ is *consecutive* if there exists an ordering of players, f, such that $S \succ_i \{i\}$ for some i implies that S is consecutive with respect to f.

A partition Π is *consecutive with respect to an ordering* f, if each $S \in \Pi$ is consecutive with respect to f.

[12] A point x is in the core of V if $x \in V(N)$ and there does not exist a nonempty S and $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$.

Theorem 1 [Adapted from Scarf (1967) and Greenberg (1994)][13] *If a game is ordinally balanced, then there exists a core stable coalition partition. If a game $(N, \succ)$ is weakly consecutive with respect to an ordering f, then there exists a core stable coalition partition that is consecutive with respect to f.*

Let us now examine the relationship between the various sufficient conditions for the existence of core stable coalition partitions. The following properties are shown to be sufficient for the existence of a core stable coalition partition by Banerjee, Konishi, and Sönmez (1998).

Given a non-empty set of players $V \subset N$, a non-empty subset $S \subset V$ is a *top-coalition* of V if for any $i \in S$ and any $T \subset V$ with $i \in T$ we have $S \succeq_i T$. A game satisfies the *top coalition* property if for any non-empty set of players $V \subset N$, there exists a top-coalition of V.

Given a non-empty set of players $V \subset N$, a non-empty subset $S \subset V$ is a *weak top-coalition* of V if it has an ordered partition $\{S^1, \ldots, S^\ell\}$ such that (i) for any $i \in S^1$ and any $T \subset V$ with $i \in T$ we have $S \succeq_i T$ and (ii) for any $k > 1$, any $i \in S^k$, and any $T \subset V$ with $i \in T$, we have $T \succeq_i S \Rightarrow T \cap (\cup_{m<k} S^m) \neq \emptyset$. A game satisfies the *weak top coalition* property if there exists a partition $\Pi = (\Pi_1, \ldots, \Pi_K)$ of N, such that for each $k \in \{1, \ldots, K\}$, Π_k is a weak top coalition of $N/(\cup_{k'<k} \Pi_{k'})$.

The following proposition outlines the relationship between these top coalition properties, and the ordinal balance and the weak consecutive properties that defined above.

Proposition 1 *A game that satisfies the top coalition property is weakly consecutive, if players' preferences are strict. The weak top coalition property, the weak consecutive property, and the ordinal balance property are completely distinct (even with strict preferences): for any given property there exists a game that satisfies the given property but fails to satisfy the other two.*

The relationship missing from the above proposition is the relationship between the top coalition property and ordinal balance. The top coalition property does not

[13] Greenberg and Weber (1986) show that consecutiveness is sufficient for core existence in NTU games. Greenberg (1994), in discussing the Greenberg and Weber result states consecutiveness in the form which corresponds to weak consecutiveness here, and correctly claims that it is sufficient for existence. The difference between these conditions is non-trivial as we shall see below that the top-coalition property implies weak consecutiveness but not consecutiveness.

necessarily imply ordinal balance, which can be seen through game 5 in Banerjee, Konishi, and Sönmez (1998).

Proof of Proposition 1: Suppose that a game satisfies the top coalition property. We show that it is weakly consecutive. Identify a top coalition of N. Call it S_1. Identify a top coalition of N/S_1, call it S_2, and so on, defining $S_3, \ldots, S_K$ in this manner. Define the ordering f, by assigning values $1, \ldots, \#S_1$ to the members of S_1 (in any order - so that $f(i) \in \{1, \ldots, \#S_1\}$ for each $i \in S_1$). Assign values $\#S_1 + 1, \ldots, \#S_1 + \#S_2$ to the members of S_2, and so on. Now, consider any Π that is blocked by some S. It must be that $\Pi \neq \{S_1, S_2, \ldots, S_K\}$, as that is clearly a core stable partition. So, find the lowest index k such that $S_k \notin \Pi$. Since S_k is a top coalition of $N/\cup_{j<k} S_j$, it follows that Π is blocked by S_k, which is consecutive under the ordering f.

The following example is of a game that is weakly consecutive, but does not satisfy the weak top coalition or ordinal balance properties. Let $N = \{1, 2, 3\}$ and

$$\begin{aligned}
&\{1,2\} \succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3\},\\
&\{1,2,3\} \succ_2 \{2,3\} \succ_2 \{1,2\} \succ_2 \{2\},\\
&\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}.
\end{aligned}$$

This is weakly consecutive when f is set by the identity, since the only coalition that is not consecutive, $\{1,3\}$, only blocks the partition $\{\{1\}, \{2\}, \{3\}\}$, which is also blocked by $\{1,2\}$.

This game does not satisfy the weak top coalition property as there is no weak top coalition of N. The only candidates are (i) $\{1,2,3\}$, which cannot be a weak top coalition since player 1 prefers $\{1\}$ and thus cannot be put in the necessary ordered partition of $\{1,2,3\}$ in the definition, and (ii) $\{1,2\}$ which cannot be a weak top coalition since player 2 can form a better coalition with player 3.

This game does not satisfy ordinal balance relative to the balanced family of coalitions $\mathcal{B} = \{\{1,2\}, \{2,3\}, \{1,3\}$. Any partition with at least on singleton player cannot make the singleton player as well off as in the coalitions in $\mathcal{B}$, and the only other partition is $\{\{1,2,3\}\}$, which leaves player 1 worse off than in the coalitions in $\mathcal{B}$.

The following game satisfies the weak top coalition property, but is not weakly consecutive and does not satisfy the ordinal balance condition. Let $N = \{1, 2, 3\}$ and

$$\{1,2,3\} \succ_1 \{1,2\} \succ_1 \{1,3\} \succ_1 \{1\},$$
$$\{2,3\} \succ_2 \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\},$$
$$\{1,3\} \succ_3 \{1,2,3\} \succ_3 \{2,3\} \succ_3 \{3\}.$$

First, we check that the game satisfies the weak top coalition property. A weak top coalition of N is $\{1,2,3\}$ with corresponding partition $S^1 = \{1\}$, $S^2 = \{3\}$, and $S^3 = \{2\}$. A weak top coalition of any V that is a strict subset of N is simply V with $S^1 = V$.

Second, we check that this game is not weakly consecutive. The only coalition that blocks $\{\{1,2\},\{3\}\}$ is $\{2,3\}$. The only coalition that blocks $\{\{1,3\},\{2\}\}$ is $\{1,2\}$. The only coalition that blocks $\{\{2,3\},\{1\}\}$ is $\{1,3\}$. There is no f for which each of these blocking coalitions is consecutive.

Third, we check that this game is not ordinally balanced. Consider the balanced family $\mathcal{B} = \{\{1,2\},\{2,3\},\{1,3\}\}$. No partition that has any player remaining single can satisfy the requirement of ordinal balance relative to this $\mathcal{B}$ since remaining single is least preferred for all players. The only possibility is then the partition $\{\{1,2,3\}\}$. However, player 2 prefers both $\{1,2\}$ and $\{2,3\}$ to $\{1,2,3\}$, and so ordinal balance cannot be satisfied.

Finally, The following game satisfies ordinal balance, but is not weakly consecutive nor does it satisfy the weak top coalition property. Let $N = \{1,2,3\}$ with preferences:

$$\{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1,3\} \succ_1 \{1\},$$
$$\{2,3\} \succ_2 \{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2\},$$
$$\{1,3\} \succ_3 \{1,2,3\} \succ_3 \{2,3\} \succ_3 \{3\}.$$

To see that the above game satisfies ordinal balance, notice that the only balanced families for which the partition $\{\{1,2,3\}\}$ does not satisfy the condition, have some pair of players in only one coalition. Thus, they are of the form $\mathcal{B} = \{\{i,j\},\{k\}\}$, which is in fact a partition.

To see that the above game is not weakly consecutive, (given the symmetry of preferences) set $f(1) = 1$. If $f(2) = 2$, then there is no consecutive blocking coalition to the partition $\{\{2,3\},\{1\}\}$, as only $\{1,3\}$ blocks. If $f(3) = 2$, then there is no consecutive blocking coalition to $\{\{1,3\},\{2\}\}$, as only $\{1,2\}$ blocks.

To see that the game fails the weak top coalition property, notice that the only candidates for a weak top coalition of N are coalitions of size 2. Without loss of generality, given the symmetry, consider the coalition $\{1,2\}$. This is not a weak top coalition since 2 can form a better coalition with 3. ∎

5 The Existence of Nash and Individually Stable Partitions

While there are hedonic settings where there exist core stable partitions, we have already seen from the example in Section 2 there are interesting settings where there are no core stable partitions. There are settings of interest where, despite the non-existence of core stable partitions, there exist coalition partitions that are immune to various sorts of individual movements.

Proposition 2 *If players' preferences are additively separable and symmetric, then an individually stable coalition partition exists. Moreover, a Nash stable coalition partition exists.*

Proof: It is clear that Nash stability implies individual stability. Therefore, we demonstrate the existence of a Nash stable coalition partition.

Let v_{ij} be a value of player i for player j, and so by symmetry $v_{ij} = v_i(j) = v_j(i)$. Player i has an incentive to move from $S_\Pi(i)$ to S_k, whenever

$$\sum_{j \in S_k} v_{ij} > \sum_{j \in S(i)} v_{ij},$$

recalling the normalization that $v_{ii} = 0$. So, consider a coalition structure Π and the sum

$$\sum_{ij : \exists S \in \Pi, ij \in S} v_{ij}.$$

If i has an incentive to move from $S_\Pi(i)$ to S_k, then by symmetry the above sum for the resulting partition Π' would be higher than that for the partition Π by $\sum_{j \in S_k} v_{ij} - \sum_{j \in S_\Pi(i)} v_{ij}$. Therefore, any coalition structure which maximizes the above sum is Nash stable. Such a maximizer exists given the finite number of possible partitions. ∎

The proof of Proposition 2 uses symmetry in a critical way. There may fail to exist a stable coalition partition if symmetry is weakened to mutuality, even when preferences are additively separable and mutual. As an easy example of non-existence of a Nash stable coalition partition with such preferences let $N = 3$ and $v_1(2) = 2$, $v_2(1) = 1$, $v_1(3) = -1$, $v_3(1) = -2$, $v_2(3) = 2$, $v_3(2) = 1$. The next example shows that an individually stable coalition partition may fail to exist even when preferences are additively separable, mutual, and single peaked on a tree. This example is adapted from one in Banerjee, Konishi and Sönmez (1998), where they show nonexistence of a core allocation.

Example 5

Let $N = \{1, \ldots, 5\}$ and have players' form a cycle such that each player likes the previous player a little, the following player a lot, and hates other players. For instance, let $v_i(i-1) = 1$, $v_i(i+1) = 2$, and $v_i(i-2) = v_i(i+2) = -4$ (where if $i = 5$ then $i+1 = 1$ and if $i = 1$ then $i-1 = 5$). Any coalition partition which includes a coalition that has players who hate each other is not individually stable as it is not individual rational. Thus, the only possible coalitions are of size one or two, where coalitions of size two must contain consecutive players. Note that any consecutive players who are alone would prefer to merge in a coalition, and so the only candidate for an individually stable coalition partition would be (without loss of generality, because of the symmetry of preferences) $\{1, 2\}$, $\{3, 4\}$ and $\{5\}$. This is not individually stable since 4 prefers to join 5, who would accept her.[14]

While the domain of additively separable and symmetric preferences is of some interest, it is a very limited one. For instance, it does not capture the setting we discussed in Section 2 surrounding Example 1. We now turn to a domain that includes the local public good setting of Example 1 as a special case. We provide a proof of the existence of an individually stable (and weakly Pareto efficient) coalition partition for this domain.

Ordered Characteristics

Each coalition $S \neq \emptyset$ is described by a characteristic or choice $c(S)$ that lies in $\{0, 1, \ldots, \#S\}$.

[14] As noted by Banerjee, Konishi and Sönmez (1998), the preferences in this example fail to be single-peaked on a tree but are easily extended to be such. Add players 6 and 7 such that: 6 likes everybody ($v_i(6) = v_6(i) = 3$ for $i \leq 5$), 7 hates everybody exept 6 ($v_i(7) = v_7(i) = -100$ for $i \leq 5$), and 6 and 7 really like each other ($v_6(7) = v_7(6) = 20$). Then any individually stable partition must contain $\{6, 7\}$, and the situation is reduced to the previous example. Preferences are single-peaked on a star-shaped tree with player 6 at the center.

Players have single peaked preferences on $\{0, 1, \ldots, n\}$, with peaks denoted p_i and $p_i \geq 1$. Their preferences over coalitions corresponds simply to the preference ranking of $c(S)$. So, if $i \in S$ and $i \in S'$, then $S \succeq_i S'$ if and only if $c(S) \succeq_i c(S')$.[15]

A hedonic game has *ordered characteristics* if players preferences over coalitions depend on single peaked preferences over coalitional choices, where the choice function $c(S)$ satisfies the following conditions:

(i) If $c(S) < \#S$, then $c(S) = p_j$ for some $j \in S$, and

(ii) If $i \notin S$, $j \notin S$, and $p_i \geq p_j$, then $c(S \cup i) \geq c(S \cup j)$. Moreover, if $c(S \cup i) > p_i$, then $c(S \cup i) = c(S \cup j)$.

Condition (i) says that if a choice is not capacity constrained, then it must choose the peak of some player. The first part of condition (ii) says that when comparing coalitions that differ by the identity of exactly one of the players, the choices of the coalitions are ordered by the peaks of the players who differ. The second part of (ii) states that the difference between these players cannot matter if a player with a peak smaller then the coalitional choice is replaced by another player with a peak not higher than hers.

Both (i) and (ii) are violated if $c(S)$ is taken as some weighted average of the peaks of coalition members (and indeed such situations fail existence), but are satisfied if it is based on some order statistic of the peaks.

Illustrative examples of settings satisfying having ordered characteristics are as follows.

(1) Players preferences are anonymous and single-peaked on the size of the coalition to which they belong.

(2) As in Section 2, players have preferences over a choice of a level of public good and $c(S) = \min[\#S, \text{median}_{i \in S}(p_i)]$, with a deterministic tie break if the coalition has an even number of members. (Any order statistic other than median, such as max or min, would also work.)

Consistency

While the ordered characteristics condition places structure on the choices of coalitions, it still allows for some inconsistencies in how those choices may be made across

[15] Haeringer (2000) explores a model where symmetric players have preferences that depend simultaneously on the size of their coalition and the choice of the coalition.

different coalitions. For instance, consider the case where the choice is equal to the full capacity of the coalition, except when there are precisely 5 players in the coalition in which case it is the minimum peak. That is, consider $c(S) = \#S$ if $\#S \neq 6$ but $c(S) = \min_{i \in S} p_i$ if $\#S = 6$. This satisfies the ordered characteristics condition, but makes life difficult in terms of finding an individually stable and Pareto efficient coalition partition (see Example 8, below)).

A hedonic game that has ordered characteristics is *consistent* if whenever there exists i and S such that $c(S \cup i) = p_i < c(S) < \min_{j \in S} p_j$, it follows that $c(T \cup i) \leq p_i$ for any T.

The consistency condition says that if some player i forces the choice of a coalition to be below the capacity constraint in a situation where the capacity constraint already falls below all of the other coalition members' peaks, then the choice of some other coalition is also no more than i's peak. While this is a very minimal sort of consistency condition, it is enough, under ordered characteristics, to guarantee the existence of an individually stable and Pareto efficient coalition partition.

Theorem 2 *If a hedonic game has ordered characteristics, then there exists an individually stable coalition partition. If, in addition, consistency is satisfied, then there exist a weakly Pareto efficient individually stable partition. Moreover, an algorithm for identifying an individually stable coalition partition that is weakly Pareto efficient is described below.*

We state the algorithm below and provide the proof of Theorem 2 in the appendix.

Before proceeding with a formal description of the algorithm, we provide some examples that illustrate some of the basic points of the algorithm. We stick to the case where players care only about the size of coalitions, and $c(S) = \#S$. The first example also shows the contrast with core stability, as it builds on Example 1 of Banerjee, Konishi, and Sönmez (1998) who show non-existence of a core stable coalition partition. As we see here, there does exist an individually stable and Pareto efficient partition.

The basic idea behind the algorithm is to start by grouping players with the highest peaks together, until we reach a size that exceeds the peak of the next player to be added. Then we start forming a new coalition, and so forth. That will not work in all cases, as we will see in Example 7 below, but serves as a starting point.

Example 6

There are seven players with the anonymous and single peaked preferences :

$\succ_1 = \succ_2$ and $4 \succ_1 3 \succ_1 5 \succ_1 6 \succ_1 2 \succ_1 1 \succ_1 7$

$5 \succ_3 4 \succ_3 3 \succ_3 6 \succ_3 2 \succ_3 1 \succ_3 7$

$\succ_4 = \succ_5 = \succ_6 = \succ_7$ and $6 \succ_4 5 \succ_4 4 \succ_4 3 \succ_4 2 \succ_4 1 \succ_4 7$

We begin by creating a first coalition by adding player 7, then 6, 5, 4, and then 3. Player 3 closes the coalition, since the coalition size is 5 and that is 3's peak. Next we form a second coalition by grouping 1 and 2 together. The resulting coalition is $\{1,2\}, \{3,4,5,6,7\}$ and it is individually stable and weakly Pareto optimal.

Note that there is an interesting implication of Theorem 2, which is illustrated in the context of the above example. The core stable existence problem arises because of instability with respect to *intermediate* sized coalitions. We know from Theorem 2 that there exist coalition partitions that are stable with respect to both changes by single players and rearrangements made by the grand coalition. In Example 6, it is a coalition of players 1 and 2 joining with 4 to 7 that upsets core stability.

We now show some adjustments that are necessary in the basic process described above to develop the algorithm.

Example 7

$p_1 = 2$ and $5 \succ_1 1$.

$p_2 = p_3 = p_4 = 3$ and $1 \succ_4 4$

$p_5 = p_6 = p_7 = p_8 = 5$

Here, beginning by grouping the players with the largest peaks we form $\{5,6,7,8\}$. Players 2, 3 and 4, prefer not to go to this largest group so we next form $\{2,3,4\}$ which gives them each their peak. Next we are left with a singleton $\{1\}$. We have to allow 1 to join $\{5,6,7,8\}$, as the partition is not individually stable as it is, and so we end up with the partition $\{\{2,3,4\},\{1,5,6,7,8\}\}$. So the algorithm should allow for players to move up to larger coalitions if they are in a coalition smaller than their peak size.

The Algorithm

Order players from 1 to n in increasing order of their peaks. So, $i \geq j$ implies $p_i \geq p_j$. Order players with the same peaks in any way.

Step 1. Form a coalition S_1 by adding player n. Next, add player $n-1$ if $p_{n-1} \geq c(\{n-1,n\})$. Continue to add players in the reverse order of their labels and add players iteratively as long as $p_k \geq c(\{k, \ldots, n\})$ and $\{k+1, \ldots, n\}$ is open to k. Stop and do not add player $k-1$ when $p_{k-1} < c(\{k-1, \ldots, n\})$ or if $\{k, \ldots, n\}$ is closed to $k-1$. Call the resulting coalition S_1. Proceed to step 2.

Step 2. Form a coalition S_2 in the same manner from the remaining players, starting with player $k-1$ who was not added in Step 1. If S_1 is closed then proceed to step 3. If S_1 is open to some players, then see if the highest indexed player in S_2 would be made better off by moving to S_1 (i.e., $c(S_1 \cup i) \succ_i S_2$). If so, and if S_1 is open to i (so $c(S_1 \cup i) \succeq_j c(S_1)$ for all $j \in S_1$) then move the player to S_1. Otherwise ask the same question of the next highest player in the S_2 and so on until either some player is moved, or all players in S_2 have been considered. If no player for whom S_1 is open would be made better off by moving, then see if there is a player who is indifferent to moving for whom S_1 is open and move the highest indexed such player. If a player is moved from S_2 to S_1, then see if there is a player i who is not yet in a coalition such that $p_i \geq c(\{S_2 \cup i\})$ and S_2 is open to i (taking S_2 in its current form - so without the player who joined S_1). If so, add the player to S_2. Iterate on this procedure, until no players are moved and no players are added to S_2. Proceed to step 3.

Step 3. Iterate on the procedure described in step 2 with the remaining players. After creating a new coalition, there may be several open coalitions when considering moving players. In that case begin with the next highest indexed open coalition. Continue the process of trying to move players as described in Step 2 (and replacing any moved player) until all players would be hurt by moving to any lower indexed coalition that is open to them. Then create a new coalition with players not yet assigned to any coalition. Stop when all players are assigned to a coalition and all players would be hurt by moving to any lower indexed coalition that is open to them.

The consistency property is non-redundant in showing the weak Pareto optimality of a partition constructed by our algorithm. The following example shows that under ordered characteristics, but without consistency, it is possible for the algorithm to find an individually stable but inefficient coalition partition.

Example 8

There are eight players. Players 1 to 5 have peak at 8, while players 6 to 8 have peak at 4 and prefer more to less. The coalition choice $c(\cdot)$ is defined as follows.

$$c(S) = \begin{cases} \#S, & \text{if } \#S \neq 6, \\ \min\{\#S; p_i, i \in S\}, & \text{if } \#S = 6. \end{cases}$$

The algorithm from Theorem 2 gives the partition $S_1 = \{1, 2, 3, 4, 5\}$, $S_2 = \{6, 7, 8\}$, while $S = \{1, \ldots, 8\}$ is Pareto superior to it.

We make some further remarks on Theorem 2. These are illustrated with examples for the special case where $c(S) = \#S$, so players care only about the size of their coalition.

Even under consistency, Theorem 2 does not imply that *all* individually stable coalition partitions are (weakly) Pareto optimal. The following example shows that this is not the case, and so the algorithm is important in selecting an individually stable coalition partition that is also Pareto optimal.

Example 9

There are eight players with anonymous and single peaked preferences over the size of their coalition. Players 1 through 4 have a peak at 2, and players 4 through 8 have peaks at 4. In this case the algorithm in the proof of Theorem 2 finds the Pareto optimal and individually stable (and core stable) coalition partition which is unique up to a relabeling of the players. It is $\{1, 2\}, \{3, 4\}, \{5, 6, 7, 8\}$.

Another individually stable coalition partition is $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$. This is Pareto dominated (in a strict sense) by the partition above.

The next example shows that Theorem 2 does not hold beyond the ordered characteristics condition. Without that condition it is possible to find preference profiles for which the individually stable coalition partitions and the (weakly) Pareto efficient ones are disjoint.

Example 10

There are four players with the following preferences:

$134 \succ_1 12 \succ_1 124 \succ_1 14 \succ_1 13 \succ_1 1234 \succ_1 123 \succ_1 1$

$12 \succ_2 124 \succ_2 23 \succ_2 1234 \succ_2 24 \succ_2 234 \succ_2 123 \succ_2 2$

$23 \succ_3 134 \succ_3 34 \succ_3 13 \succ_3 1234 \succ_3 234 \succ_3 123 \succ_3 3$

$134 \succ_4 34 \succ_4 124 \succ_4 14 \succ_4 24 \succ_4 1234 \succ_4 234 \succ_4 4$

It is easily checked that $\{1, 2, 3, 4\}$ is the unique individually stable partition. However, it is not Pareto optimal, since everybody prefers the partition $\{1, 2\}, \{3, 4\}$.

Next, we show that Theorem 2 does not extend to Nash stability. There may not exist a Nash stable coalition partition, even if preferences are anonymous and single-peaked over size of coalition. A trivial example can make this point: have two individuals with $\{1,2\} \succ_1 \{1\}$ and $\{2\} \succ_2 \{1,2\}$. The following example shows that this is also true in less degenerate cases.

Example 11

Consider $N = \{1,2,3,4\}$, and the anonymous, single-peaked preferences over coalition sizes described by

$4 \succ_1 3 \succ_1 2 \succ_1 1$,

$3 \succ_2 2 \succ_2 1 \succ_2 4$,

$2 \succ_3 3 \succ_3 1 \succ_3 4$, and

$2 \succ_4 3 \succ_4 1 \succ_4 4$.

There does not exist a Nash stable coalition partition:

– the partition $\{\{1,2,3,4\}\}$ is not Nash stable as player 4 prefers to be alone;

– a partition of the form $\{\{a,b,c\},\{d\}\}$ is not Nash stable as players 3 and 4 like size 2 better than size 3 and one of them must belong to $\{a,b,c\}$ and thus would prefer to leave and join d;

– a partition of the form $\{\{a,b\},\{c,d\}\}$ is not Nash stable as player 1 would like to switch coalitions since she prefers size 3 to size 2;

– a partition of the form $\{\{a,b\},\{c\},\{d\}\}$ is not Nash stable as player c (regardless of identity) would prefer to join the coalition $\{a,b\}$;

– the partition $\{\{1\},\{2\},\{3\},\{4\}\}$ is not Nash stable as any player prefers to form a couple with another player. ▮

It is interesting to remark that unlike many models where single-peakedness is postulated (e.g., median voting in public goods environments and uniform allocations in allotment problems), we need information about players' preferences beyond knowing what their peaks are in order to construct a stable coalition partition. This is demonstrated in the following example.

Example 12

Let $n = 4$, $p_1 = 2$, and $p_i = 4$ for $i \in \{2,3,4\}$.

If player 1 prefers size 4 to size 1, then $\{\{1,2,3,4\}\}$ is the only individually stable coalition partition.

If player 1 prefers size 1 to size 4, then $\{\{1\},\{2,3,4\}\}$ is the only individually stable coalition partition.

Next, we show that single-peakedness is important in establishing Theorem 2. Although it can be checked that for $n \leq 7$ when players care only about coalition size, anonymity alone suffices for the existence of an individual stable coalition partition, this is not true for larger n. The following example with $n = 63$ is the smallest example that we know of where anonymity of preferences does not suffice for the existence of an individually stable coalition partition.

Example 13

Let $n = 63$. Anonymous preferences over coalition size are described below, where the argument is the size of the coalition.

$57 \succ_1 2 \succ_1 7 \succ_1 6 \succ_1 1 \succ_1$ remaining sizes.

$7 \succ_2 2 \succ_2 57 \succ_2 56 \succ_2 1 \succ_2$remaining sizes.

$2 \succ_3 7 \succ_3 6 \succ_3 1 \succ_3$ remaining sizes.

For players 4 to 8: $7 \succ_4 6 \succ_4 5 \succ_4 4 \succ_4 3 \succ_4 2 \succ_4 1 \succ_4$ remaining sizes.

For players 9 to 63: single peaked with peak at 63.

We verify that there is no individually stable partition.

Claim 1. Any candidate individually stable coalition structure must have players 9 to 63 together (possibly with others, too). Call this S_1.

Proof of Claim: Note from the preferences that any coalition that is closed (and that is preferred to staying alone by all its members) must be of size 7 or less or of size 57. Also note that players indexed from 9 up prefer larger to smaller, so they must all be in the largest coalition if one of the largest coalitions is open, and then this must be uniquely the largest. Suppose the claim to be false. Then it must be that all of the largest coalitions are closed. If the largest coalition has 57 members, then it must not contain any of players 3 to 8 and so contains players 9 to 63 as claimed. So it must be that all the largest coalitions are closed and of size 7 or less. Any closed coalition of size 7 or less must have a player from 1 to 8 in it. There are at most 8 such coalitions which takes up at most 56 players. Thus there are at least 7 remaining players who are indexed from 9 to 63 and are together in an open coalition, which thus must have maximal size, which leads to a contradiction.

Claim 2. Any candidate individually stable coalition structure must have players 4 to 8 together (possibly with others, too). Call this S_2.

As S_1 has at least 55 members, there can be at most eight remaining players in the other coalitions. None of players 4 to 8 can be in S_1, as they would rather be alone. From the preferences it follows that any of the coalitions other than S_1 that are closed (and that are preferred to staying alone by all its members) must be of size 2 or 7. If none of the largest coalitions other than S_1 are closed, then it must be that all of player 4 to 8 are in it, since they prefer larger to smaller up to a size of 7. If there is a closed coalition of size 7, and the claim were not true, then there must be a player indexed 4 to 8 who is alone. However, this could not be as then player 3 would like to join this single player. Thus, for the claim to be false, it must be that all of the largest coalitions other than S_1 are closed, and they must all be of size 2 or less. Each such coalition must have a player indexed 1, 2, or 3 in it and so there are at least 2 players indexed 4 to 8 left. They would be together in an open coalition which is a contradiction.

Claim 3. At most two of players 1, 2, and 3 are in the same coalition.
If all 3 players were in the same coalition then from Claims 1 and 2 it follows that that coalition would be of size 58, 8, or 3. In any one of these cases, any of the three players would prefer to be alone.

Claim 4. At least two of players 1, 2, and 3 are in the same coalition.
From Claims 1 to 3 it follows that if they were all separated, then at least one of them would have to be alone. From the preferences of player 3, it would have to be only player 3, or else he would join the single player. Thus, S_1 has 56 members, S_2 has 6 members, and player 3 is alone. This is not stable, as both players 1 and 2 prefer 7 members to 56, so whichever one of them is in S_1 should join S_2.

Claims 3 and 4 tell us that exactly two of players 1,2, and 3 are in the same coalition. Let us now consider separate cases.

Case 1: 1 and 2 are both in the same coalition.

If 1, 2 are not in S_2, then it must be that 3 is in S_2. But then player 2 would like to join S_2 and would be accepted, which violates individual stability. If 1, 2 are in S_2, then 3 is alone, but 1 would like to join 3 and would be accepted, which violates individual stability.

Case 2: 2 and 3 are both in the same coalition.

If 2, 3 are not in S_2, then it must be that 1 is in S_2. But then player 2 would like to join S_2. Contradiction. If 2, 3 are in S_2, then 1 is alone. But then 3 would like to join 1. Contradiction.

Case 3: 1 and 3 are both in the same coalition.

If 1, 3 are not in S_1, then it must be that 2 is in S_1. But then player 1 would like to join S_1. Contradiction. If 1, 3 are in S_1, then 2 is alone. But then 3 would like to join 2. Contradiction.

The above results and examples show that there are plausible situations where individually stable coalition partitions exist, but also others where they do not. Let us briefly examine a less restrictive notion of local stability: contractual individual stability. This has the nice property of always existing, but suffers from strong assumptions on the limits of mobility that players have. Also, as the following Proposition demonstrates, contractual individual stability has an interesting relationship with Pareto efficiency, which follows the reasoning of Dreze and Greenberg (1980), in a different context.

Proposition 3 *Any Pareto efficient coalition partition is contractually individually stable. If preferences are strict*[16]*, then there exists a Pareto efficient and individually rational coalition partition that is contractually individually stable.*

Proof: We first show that any Pareto optimal coalition partition, Π, is contractually individually stable.

Consider any i, $S_\Pi(i)$, and $S_k \in \Pi \cup \emptyset$, where $S_\Pi(i) \neq S_k$. Consider Π' where the only change from Π is that i moves from $S_\Pi(i)$ to S_k. By the Pareto optimality of Π, it follows that either all players are indifferent between the two partitions, or that some player j is worse off under the new partition. Since preferences are hedonic, it must be that $j \in S_\Pi(i) \cup S_k$. In either case, this change for i would not be viable under the definition of contractual individual stability. Since these choices were arbitrary, Π is contractually individually stable.

Next, we show that if preferences are strict then the following algorithm identifies a Pareto efficient and individually rational coalition partition that is contractually individually stable.

Start with player 1. Let $\widehat{S}_1$ be the best coalition for player $i_1 = 1$ subject to the constraint that no player in the coalition would prefer to be alone. So $\widehat{S}_1 = \max_{\succ_1}\{S \subset N : 1 \in S \text{ and } S \succeq_j \{j\}\ \forall j \in S\}$. Next, let i_2 be the player with the lowest index among $N - \widehat{S}_1$, and define $\widehat{S}_2$ to be the best coalition for player i_2 among subsets of $N - \widehat{S}_1$ such that no player in the coalition would prefer to be alone; etc. Continuing in this way, define i_k and $\widehat{S}_k$ accordingly. It follows from the definition of the algorithm that the partition $\left\{\widehat{S}_k\right\}$ is individually rational. The partition is contractually individually stable by the following reasoning. Individual $i_1 = 1$ is in her most preferred coalition subject to individual rationality, so she will prevent anyone from leaving it. The only

[16] A player has a strict preference over any two coalition partitions for which her coalition differs.

players she would admit to S_1 are ones that would rather remain alone. Given this, similar reasoning applies to each i_k and S_k, successively. A similar argument proves Pareto efficiency: adding any players to $\widehat{S}_1$ would make either some added player or player i_1 worse off. Subtracting players from $\widehat{S}_1$ would make player i_1 worse off. Thus, a Pareto improvement must leave $\widehat{S}_1$ unchanged. Similar reasoning applies to each $\widehat{S}_k$, successively. ∎

6 Concluding Remarks

We have focussed on the existence of coalition partitions that are stable to the movements of one player at a time, and noted some relationships between various forms of such stability and Pareto efficiency. While existence of stable coalition partitions may be reassuring, one also cares about the properties that the stable coalition partitions will exhibit. Will they be fair, for example, treating equal players equally and being envy free? How will they adjust as the population grows? Will players have incentives to misrepresent their preferences when forming a coalition? These are important questions to address in further research.

To close, we provide a few examples that suggest that one will have to identify special domains in order to make sure that stable coalition partitions are nicely behaved.

First, we point out that there are some very simple situations where stable partitions will necessarily treat players asymmetrically. For example, suppose that a society contains three players who care only about coalition size. They all have preferences over coalition sizes being described by $2 \succ_i 1 \succ_i 3$. Here, any individually stable coalition partition (as well as core stable, Nash stable, and individually contractually stable partition) consists of two players together and one player alone. This violates most notions of fairness.[17] [18]

Of course, fairness in a traditional sense of full symmetry, like equal treatment of equals or envy-freeness, is generally difficult (if not impossible) to achieve in settings with indivisibilities. In our case though, fair allocations exist at the expense of efficiency

[17] For example, it violates envy-freeness and equal treatment of equals. Say that a partition is envy free if $S_\Pi(i) \succeq_i (S_\Pi(j) \cup i) \setminus j$ for every i and $j \notin S_\Pi(i)$. Say that it satisfies equal treatment of equals if $S_\Pi(i) \sim_i (S(j) \cup i) \setminus j$ for every i and j such that $\succ_i$ and $\succ_j$ are the same (under a permutation of i and j).

[18] The example also shows that Pareto efficiency is incompatible with fairness. This suggests that it is necessary to sacrifice symmetry and fairness in order to achieve either individual stability or efficiency. This is not surprising in a setting with such indivisibilities, but is noteworthy nonetheless.

and stability. One can simply place all players in one coalition, or leave each one alone, or divide them in groups of the same size. Another common way to restore fairness, is to allow for some randomization, and to take an ex ante perspective, which could be employed here.

It is worth pointing out that in our setting we can regard stability as a requirement that bears some "restricted fairness" flavor. Indeed, Nash stability guarantees that any player will not be interested in joining another group. This is a weakening of an envy-freeness requirement, which demands that any player will not be interested in replacing a player in another group.

Next, we point out that the set of individual stable partitions do not necessarily evolve nicely as the population size changes. Consider $N = \{1, 2, 3, 4\}$, and the anonymous preferences that are single-peaked over coalition size represented by

$2 \succ_1 1 \succ_1 3 \succ_1 4$ and

$4 \succ_i 3 \succ_i 2 \succ_i 1$, for $i \in \{2, 3, 4\}$.

Start with just players 1 and 2. The unique individually stable coalition partition is $\{\{1, 2\}\}$. Next, add players 3 and 4. The unique individually stable coalition is $\{\{1\}, \{2, 3, 4\}\}$. Player 1 is worse off and player 2 is better off due to the arrival of the new players. This violates population monotonicity, which says that players welfare should move in the same direction due to a change in population size (for example, see Thomson (1991)).

We conclude with an example showing the incentives of players to misrepresent their preferences in situations where preferences are be private information. Take n to be even and $n \geq 4$. (Slight variations on this example work when n is odd.) If all players have peak $n - 1$ and find n least preferred, then the only individually stable partitions are splits into two coalitions of sizes 1 and $n - 1$. Consider, any such partition, which without loss of generality we take to be $\{1\}, \{2, \ldots, n\}$. Let all players except n have the same preferences as before, and the preferences of n change to have a peak at 2, with size 1 being least preferred. It is easy to check that here the only individually stable partition is $\{n\}, \{1, \ldots, n-1\}$. Note that in the second situation, player n would be better off pretending to have a peak of $n - 1$.

Thus, for $n \geq 4$, there does not exist a non-manipulable rule on the domain of anonymous and single-peaked over coalition size preferences, which would always select an individually stable partition. This suggests difficulties in combining non-manipulability of the rule and stability of outcomes.

The above discussion points out that in addition to further study of the existence of stable and efficient coalition partitions in the hedonic model, there is a rich set of questions to be analyzed regarding the satisfaction of other desirable properties.[19]

[19]Some recent papers provide further analysis in these directions. For instance, see Alcaldé and Revilla (1999), Alcaldé and Romero-Medina (2000), Burani and Zwicker (2000), Haeringer (2000), Milchtaich and Winter (1997), and Papai (2000).

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Appendix

Proof of Theorem 2: To prove that the algorithm results in an individually stable partition we establish the following claims.

Claim 1 At any point in the algorithm, for any coalition S that is partially or fully formed, $\max_{i \in S} p_i \geq c(S)$.

If S is a singleton, then this is obvious. Otherwise, let k be the last player who was added to S. Suppose the contrary of the claim. By (i) in the definition of ordered characteristics, it follows that $c(S) = \#S > \max_{i \in S} p_i$. If $c(S \setminus k) = \#S - 1$, then some i in S would have closed S to k's entry. So it must be that $c(S \setminus k) \neq \#S - 1$, and by (i) it follows that $c(S \setminus k) = p_j$ for some $j \in S \setminus k$. Then j should have closed $S \setminus k$ to k which is a contradiction.

Claim 2 The only way $c(S)$ may decrease for some S at some point in the algorithm is for a player to leave S. So, once S_{k+1} has started to be formed, $c(S_k)$ can only increase.

This follows from (the proof of) Claim 1.

Claim 3 For any k, $c(S_k) \geq \max_{i \in S_{k+1}} p_i$.

Suppose the contrary of the claim. It must be that $p_i > c(S_k)$ for some $i \in S_{k+1}$. This means that this must be true for the highest indexed player who was not in a coalition when S_{k+1} was starting to be formed. As this player has a p_i that is no higher than all those in S_k at that time (call it S'_k), it follows that $p_j > c(S_k) \geq c(S'_k)$ for all $j \in S'_k$. This means that $c(S'_k) = \#S'_k$ and $c(S'_k \cup i) = \#S'_k + 1$, which is no more than p_i. Then by (i) and Claim 2, it follows that i would have been added to S_k under the algorithm when S_k was being formed. This is a contradiction.

Claim 4 $c(S_k) \geq c(S_{k+1})$ for all k.

This follows from Claims 1 and 3.

Claim 5 At any point in the algorithm, if some $S_k \neq \emptyset$ is open to a player $i \notin S_k$, then $c(S_k) + 1 \geq c(S_k \cup i) \geq c(S_k)$.

If S_k is open to a player i, then by Claim 1 it follows that $c(S_k \cup i) \geq c(S_k)$. If $c(S_k \cup i) > c(S_k) + 1$ then it must be that $c(S_k) \neq \#S_k$ and so by (i) of the definition of ordered characteristics there exists $j \in S_k$ such that $p_j = c(S_k)$. But then S_k will be closed to i.

Claim 6 For any nonempty S at any point in the algorithm, if $i \notin S$, $l \notin S$, $c(S) \geq p_i \geq p_l$, and S is closed to i, then S is also closed to l. After S is formed and we proceed to form the next coalition in the algorithm, this statement also becomes true for $i \notin S$, $l \notin S$, $c(S) \geq p_l > p_i$.

Suppose that S is closed to i but open to l. Then it must be that $c(S \cup i) \neq c(S \cup l)$ and so (ii) in the definition of ordered characteristics implies that $p_i > c(S \cup l)$. This implies that $c(S) > c(S \cup l)$, which by Claim 1 implies that S should have been closed to l which is a contradiction. If $p_l > p_i$, then from (ii) we have $c(S \cup l) \leq p_l$ and so l (or, by the first part of the claim, some other player with at least as high a peak) would have been added to S during the time when S was formed.

Claim 7 When S_k is being formed, $c(S_k)$ does not increase if i is moved from S_k to S_l, $k > l$, i.e. $c(S_k - i) \leq c(S_k)$.

Let j be the last player added to S_k before i is moved. We have that $c(S_k - j) \leq c(S_k)$. Since $p_j \leq p_i$, (ii) implies that $c(S_k - i) = c((S_k - i - j) \cup j) \leq c((S_k - i - j) \cup i) = c(S_k - j) \leq c(S_k)$.

Claim 8 $c(S_k) \leq p_j$ for all $j \in S_k$ except if j was moved to S_k from some S_ℓ with $\ell > k$.

This is true when j enters by the definition of the algorithm, If $p_j > c(S_k)$ at the time a new player l is admitted it follows from Claim 5 that $p_j \geq c(S_k \cup l)$. If $p_j = c(S_k)$ and a new player l is admitted then it must be that $c(S_k) = c(S_k \cup l)$. Finally, by Claim 7 the desired inequality remains true when some player is moved from S_k.

Claim 9 After a first player i is moved from S_k, $c(S_k)$ cannot grow beyond the value it had just before i left.

By Claims 7 and 8 it follows that $c(S_k - i) \leq c(S_k) \leq \min_{j \in S_k} p_j$ (in particular, $c(S_k - i) \leq p_i$). If $c(S_k - i) \in \{p_j | j \in S_k - i\}$, then it will not change if a new players is added, or j would block such an addition. Suppose that $c(S_k - i) < \min_{j \in S_k - i} p_j$, and hence $c(S_k - i) = \#S_k - 1$ and either $c(S_k) = c(S_k - i) = \#S_k - 1 = p_i$ or $c(S_k) = \#S_k \leq \min_{j \in S_k} p_j$. Suppose that some player x is added to $S_k - i$. By (ii), $c(S_k - i \cup x) \leq c(S_k - i \cup i) = c(S_k)$. If $p_x \leq c(S_k - i \cup x)$, then the choice $c(\cdot)$ of this coalition will not change subsequently. If $c(S_k - i \cup x) < p_x$, then $c(S_k - i \cup x) = \#S_k = c(S_k) < p_x \leq \min_{j \in S_k} p_j$. This implies that $c(S_k + x) = \#S + 1 \leq \min_{j \in S_k \cup x} p_j$, and so x (or, by Claim 6, some other player with a peak at least as high) would be added to S_k before i was moved.

Claim 10 After some player i is moved to S_k from a coalition with higher number for the first time, the choice of the coalition will not change (if any other players are also added).

By Claim 8 $c(S_k) \leq \min_{j \in S_k} p_j$. Thus, the only situation where $S_k \cup i$ can be open to new players who change its choice is if $c(S_k \cup i) \leq \min_{j \in S_k \cup i} p_j = p_i$. But in this case i (or some other player with a peak at least as high) would be added to S_k when it was originally formed.

Let us now verify that the resulting coalition partition is individually stable.

Consider the incentives of a player $i \in S_k$ to move to some open $S_{k'}$ where $k' < k$. By Claims 3, 4 and 5, it follows that $c(S_{k'} \cup i) \geq c(S_{k'}) \geq p_i > c(S_k)$. By the definition of the algorithm and Claim 2 it follows that $c(S_{k'} \cup i) > p_i > c(S_k)$, or else i (or some other player with at least as high a peak, given (ii)) would have been added to $S_{k'}$ during the step where $S_{k'}$ was formed. Consider the last time that i was offered a chance to move to a larger coalition. She did not prefer $S_{k'}$ over S_k at that time, and nobody from S_k was moved to a lower indexed coalition after that. By Claim 2, neither $c(S_k)$ nor $c(S_{k'})$ could have decreased since that point. So, since i did not prefer $S_{k'}$ over S_k when she got the last chance to move, then she will not want to move in the final partition.

Next, consider the incentives of a player $i \in S_k$ to move to some open $S_{k'}$ where $k' > k$. We need only consider i such that $p_i < c(S_k)$, as it follows from Claims 1 and 5 that for $S_{k'}$ to be open to i it must be that $\max_{j \in S_{k'}} p_j \geq c(S_{k'} \cup i)$; which by Claim 3 implies that $c(S_k) \geq c(S_{k'} \cup i)$. This implies (by Claim 8) that i was a player who moved into S_k from some other coalition S_l, $l > k$. Hence by Claim 10 $c(S_k)$ did not change after she moved in. Let us consider three cases on the ordering of k' and l, and show that in each case i could not benefit from moving.

If $k' < l$, then $S_{k'}$ was closed for i at the time when i moved to S_k (if it would be open we would have $c(S_{k'} \cup i) \leq \max_{p_j \in S_{k'}} p_j \leq c(S_k)$ and i would move to $S_{k'}$). But then by Claim 6 $S_{k'}$ is also closed for other players from any $S_{l'}$, $l' > k'$, so it will not change and thus will not become open for i later.

If $k' > l$, then $c(S_{k'}) \leq c(S_l)$, and since $c(S_l)$ did not grow from the time i left it (call S'_l the composition of S_l just before i left), we have that $c(S_{k'}) \leq c(S_l) \leq c(S'_l)$. Suppose that $S_{k'}$ is open to i. From Claim 5, $c(S_{k'} \cup i) \in \{c(S_{k'}), c(S_{k'}) + 1\}$. Player i could be willing to move to $S_{k'}$ only if $c(S'_l) < c(S_{k'} \cup i)$, which is only possible when $c(S_{k'}) = c(S_l) = c(S'_l) = c(S_{k'} \cup i) - 1$. But as $c(S_{k'})$ changes by admitting i it must be that $c(S_{k'}) < \min_{j \in S_{k'}} p_j$ and hence $c(S_l) < \min_{j \in S_{k'}} p_j$ which contradicts Claim 3.

If $k' = l$, then using the same reasoning as above (case $k' > l$) we obtain that i could be willing to move to S_l only if $c(S_l) = c(S'_l) = c(S_l \cup i) - 1$, and that it must be that $c(S_l) < \min_{j \in S'_l} p_j \leq \min_{j \in S_l} p_j$ (as S_l is obtained from S'_l after some players left and some with lower indices were added). This implies that $c(S_l) = c(S'_l) = \#S'_l = \#S_l$.

Consider now the player $x \in S_l - S'_l$ with the highest index. Since $c(S'_l) = \#S'_l$, we have that $c(S'_l \cup x) \leq \#S'_l + 1 = c(S'_l) + 1 \leq p_x \leq \min_{j \in S'_l} p_j$, i.e. $c(S'_l \cup x) = \#S'_l + 1$. But then x (or some other player) should have been added to S'_k before moving i.

Next, we show that any ending coalition partition under the algorithm is weakly Pareto optimal whenever the consistency condition (iii) is satisfied. Suppose the contrary, and consider a partition $\Pi = \{S_k\}_{k=1}^{K}$ which is the stopping point of the algorithm, but is not weakly Pareto efficient, where the indexing of Π corresponds to the order in which coalitions were formed in the algorithm. Let $\mathcal{T} = \{T_k\}_{k=1}^{K'}$ be a Pareto improvement of Π, such that $S_{\mathcal{T}}(i) \succ_i S_{\Pi}(i)$ for all i for whom $S_{\Pi}(i) \neq S_{\mathcal{T}}(i)$. The following observations are helpful in reaching a contradiction.

1. If $S_k = S_{\Pi}(i) \neq S_{\mathcal{T}}(i)$ for some i, then $c(S_k) = \#S_k$.

This follows from our definition of weak Pareto improvement and property (i) of ordered characteristics.

2. $c(S_k) =\geq \max_{i \in S_{k+1}} p_i > c(S_{k+1}) = \#S_{k+1}$ for each k such that there is $i \in S_{k+1}$ with $S_{\Pi}(i) \neq S_{\mathcal{T}}(i)$.

This follows from Claims 1 and 3 and observation 1.

3. If $S_k = S_{\Pi}(i) \neq S_{\mathcal{T}}(i)$ for some i, then either $c(S_k) < \min_{i \in S_k} p_i$ or there exists a unique player $j \in S_k$ such that $p_j < c(S_k) < \min_{i \in S_k - j} p_i$.

Observation 1 implies that $c(S_k) \neq p_i$ for all $i \in S_k$. By Claim 8, $c(S_k) > p_j$ for some $j \in S_k$ is possible only if j was moved to S_k after it was initially formed. Consider such a j who was moved to S_k first, and let S'_k be the composition of S_k just before j joined it. By Claim 10, the choice of S_k subsequently remains equal to $c(S'_k \cup j)$, and thus to $\#S_k = c(S_k)$ (by observation 1). If some players were moved to S_k after j then from property (i) we have that $\#(S'_k \cup j) < \#S_k = c(S'_k \cup j)$, which is a contradiction.

Index the k's such that $S_k = S_{\Pi}(i) \neq S_{\mathcal{T}}(i)$ for some i by $k_1, \ldots, k_\ell$ preserving the ordering so that $k_h < k_j$ when $h < j$. In what follows, we substitute h, $h+1$, and h' for k_h, k_{h-1}, and $k_{h'}$, whenever convenient.

4. For each $h \geq 2$ there exists $T_h \in \mathcal{T}$, $T \neq S_{\Pi}(i)$ for any i, such that $\#S_h = c(S_h) < c(T_h) \leq c(S_{h-1})$, and there exists $T_1 \in \mathcal{T}$ such that $c(S_{k_1}) < c(T_1)$.

By observation 2, it follows that there exists $T_1 \in \mathcal{T}$ such that $c(S_{k_1}) < c(T_1)$. So suppose the contrary of the observation, and find the smallest $h \geq 2$ such that 4 does not hold. Observation 2 implies that there exists $i \in S_h$ for whom $c(\mathcal{T}(i)) > c(S_h)$. Then by our supposition, $c(\mathcal{T}(i)) > c(S_{h-1})$. Since $c(\mathcal{T}(i)) \succ_i c(S_h)$ and $c(\mathcal{T}(i)) > c(S_{h-1})$, it follows from Claim 4 that each S_k with $k_h > k \geq k_{h-1}$ must be closed to i, or i would have been moved under the algorithm.

Case 1. $\min_{j \in S_{h-1}} p_j > c(S_{h-1})$.

This implies that it must be that $c(S_{h-1} \cup i) = p_i < c(S_{h-1})$, or else S_{h-1} would not be closed to i. By consistency (iii) it follows that $c(\mathcal{T}(i)) \leq p_i$ which contradicts the fact that $c(\mathcal{T}(i)) > c(S_{h-1}) \geq p_i$.

Case 2. $\min_{j \in S_{h-1}} p_j \leq c(S_{h-1})$.

By observation 3 it follows that $\min_{j \in S_{h-1}} p_j < c(S_{h-1})$. This implies that j was moved to S_{h-1} from some S_k with $k > k_{h-1}$. By the reasoning above and condition (ii), it also follows that j was offered the opportunity to move to S_{h-1} before i was and so $k_h \geq k$. Since $c(\mathcal{T}(j)) \succ_j c(S_{h-1})$, this implies by Claim 9 that $c(\mathcal{T}(j)) > c(S_k)$, which by Claim 4 implies that $c(\mathcal{T}(j)) > c(S_h)$. Thus, $c(S_h) < c(\mathcal{T}(j)) \leq c(S_{h-1})$, which contradicts our supposition.

Finally, a contradiction is reached from observation 4 which implies that the population comprising $\mathcal{T}$ must be larger than that comprising Π. ∎